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# ON SEMICOMMUTATIVE Π-REGULAR RINGS.

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**Introduction.** Let K be a ring with 1 and $a \in K$. If for every $b \in K$ there exist $r, h \in K$ such that $ab=ra$ and $ba=ah$, then a is called semicommutative. If every element of K is semicommutative, then K is called a semicommutative ring. Observe that every commutative ring is semicommutative. Let S be a direct sum of two noncommutative division rings. Then S is an example of a semicommutative ring which is not commutative. The ring K is called $\pi$-regular [1] if and only if for every $x \in K$ there exist $n \geq 1$ and $y \in K$ such that $x^n y x^n = x^n$ and unit $\pi$-regular ring iff for every $x \in K$ there exist $m \geq 1$ and a unit u of K such that $x^m u x^m = x^m$.

Let R be a semicommutative ring. In this paper, we show that all $\pi$-regular rings are unit $\pi$-regular rings and the set Nil(R) of all nilpotent elements of R is a two-sided ideal of R. We show that R is a $\pi$-regular ring iff R/Nil(R) is regular. Moreover, we show that if 2 is a unit in a $\pi$-regular ring R, then every element of R is a sum of two units in R.

**Notations.** Let S be a ring with 1. Then

1. Id(S) denotes the set of all idempotents of S.
2. Nil(S) denotes the set of all nilpotent elements of S.

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3. U(R) denotes the set of all units of S.

4. C(S) denotes the center of S.

5. J(S) denotes the Jacobson radical of S.

6. If $a,b\in S$, then (a,b) denotes the two-sided ideal of S generated by a,b, and (a,b)S denotes the right ideal of S generated by a,b, and S(a,b) denotes the left ideal of S generated by a,b.

7. Let I be an ideal of S and $x\in S$. Then [x] denotes the element x+I in R/I.

8. Let $z\in S$. Suppose for some $x,y\in S$, z is written as a product of powers of x and y. Then we define E(z,x) to be the sum of exponents of x as they appeared in z, and E(x,y) to be the sum of exponents of y as they appeared in z. For example, if $z=x^3yx^2y^3$, then E(z,x)=5 and E(z,y)=4.

Through out this paper, the capital letter R denotes a semicommutative ring with 1.

**Lemma 1.** The set Nil(R) is a two-sided ideal of R.

**Proof.** Let $x,y\in Nil(R)$. Then for some $n,m\geq 1$, $x^n=y^m=0$. Consider the expansion of $(x+y)^{m+n}$. Let z be a term in the expansion. Then E(z,x)+E(z,y)=n+m. suppose $E(z,x)=k\geq n$. Since R is semicommutative, for some $d\in R$ we have $z=x^kd=0$. Suppose $E(z,y)=h<n$. Then $E(z,y)=w\geq m$. Since R is semicommutative, for some $f\in R$ we have $z=y^wf=0$. Thus $(x+y)^{m+n}=0$. Let $q\in Nil(R)$, and $a,b\in R$. Then for some $r\geq 1$, $q^r=0$. Since R is semicommutative, $(aqb)^r=q^re=0$ for some $e\in R$. Thus Nil(R) is a two-sided ideal of R.

**Lemma 2.** The set Id(R) is a subset of C(R).

**Proof.** Let $e\in Id(R)$ and $x\in R$. Then for some $y,z\in R$, ex=ye and xe=ez. Hence exe=ye=ex and exe=ez=xe. Thus ex=xe and the proof is completed.

Before stating the first major result, the following well-known Lemma is needed.

**Lemma 3.** Let K be a ring with 1 and I be a two-sided nil ideal of R. If $[c]\in Id(K/I)$, then there exists $e\in Id(K)$ such that $[e]=[c]$ in K/I.

**Proof.** see [9].

**Theorem 1.** The ring R is $\pi$-regular if and only if R/Nil(R) is regular.

**Proof.** Suppose R is $\pi$-regular. Let $[x]\in R/Nil(R)$. Then for some $y\in R$ and $n\geq 1$, $x^n y x^n = x^n$. Thus $e=x^n y\in Id(R)$ and therefore $1-e\in Id(R)$. Since $1-e\in C(R)$, $((1-e)x)^n=(1-e)x^n=(1-x^n y)x^n=0$. Thus $(1-e)x = (1-x^n y)x \in Nil(R)$. Thus $[x][x^{n-1}y][x]=[x^n y][x]=[x]$.

Suppose $K=R/Nil(R)$ is regular. Let $x\in R$. Then for some $[y]\in K$, $[x][y][x]=[x]$ in K. Thus $[xy]\in Id(K)$. By Lemma 3, there exists $e\in Id(R)$ such that $[e]=[xy]$ in K. Thus $[e][x]=[x]$ in K. Hence $[e][x][y]=[xy]=[e]$ in K. Thus for some $w\in Nil(R)$, $exy-e=w$ and therefore $exy=e+w$ in R. Since $Nil(R)\subset J(R)$, $v=1+w\in U(R)$. Hence $exy+1-e=v$. Let u be the multiplicative inverse of v. Then $exyu+(1-e)u=1$. Thus $e(exyu+(1-e)u)=e$. Hence $exyu=e$. Since $[e][x]=[x]$ in K, $(1-e)x\in Nil(R)$. Thus for some $m\geq 1$, $((1-e)x)^m=(1-e)x^m=0$. Hence $x^m=ex^m$. Since $exyu=e$, $(exyu)^m=e^m=e$. Since R is semicommutative, for some $d\in R$ we have $(exyu)^m=(ex)^m d=e$. Since $e\in C(R)$, $(ex)^m d=ex^m d=e$. Since $ex^m=x^m$, we have $x^m d x^m=x^m$. Thus R is $\pi$-regular.

In the following theorem, we show that if for some ring K with 1 such that $Id(K)\subset C(K)$ and $x\in K$ is regular, then x is unit regular.

**Theorem 2.** Let K be a ring with 1 such that $Id(K)\subset C(k)$ and $x\in K$ is regular, then x is unit regular.

**Proof.** If $xyx=x$, then $xy, yx\in Id(K)\subset C(K)$. Hence, $xy=x(yx)y=(xy)(yx)=y(xy)x=yx$. Let $u=x+xy-1$ and $v=xy+xy^2-1$. Since $xy=yx$ and $xyx=x$, we have $uv=vu=x^2y+x^2y^2-x+(xy)^2+xyxy^2-xy-xy-xy^2+1=x+xy-x+xy+xy^2-xy-xy-xy^2+1=1$. Moreover, $xvx=x^2yx+x^2y^2x-x^2=x^2+x-x^2=x$.

A consequence of the above Theorem is the following Corollary.

**Corollary 1.** Let K be a $\pi$-regular ring with 1 such that $Id(K)\subset C(K)$. Then K is unit $\pi$-regular. In particular, if R is $\pi$-regular, then R is unit $\pi$-regular.

**Proof.** Let $x\in K$. Then for some $n\geq 1$, $x^n$ is regular. Hence, by Theorem 2, $x^n$ is unit regular. If R is $\pi$-regular, then by Lemma 2 we have $Id(R)\subset C(R)$ and therefore the claim is evident.

Let $a,b,x\in R$. Since R is semicommutative, $axb=xd=fx$ for some $d,f\in R$. Thus $(a,b)=(a,b)R=R(a,b)$. In particular, if $(a)=R$, then $a\in U(R)$. Also, observe that if $[u]\in U(R/Nil(R))$, then for some $[v]\in R/Nil(R)$, $[u][v]=[1]$ and hence $uv=1+z$ for some $z\in Nil(R)$. Since $Nil(R)\subset J(R)$, $1+z\in U(R)$ and therefore $(u)=R$. Thus $u\in U(R)$. Hence $[u]\in U(R/Nil(R))$ if and only if $u\in U(R)$. Hence we have the following Lemma.

**Lemma 4.** Let $K=R/Nil(R)$ and $u\in R$. Then $[u]\in U(K)$ if and only if $u\in U(R)$.

The second major result has to do with the structure of a $\pi$-regular ring R. We begin with the following Lemma.

**Lemma 5.** Let K be a unit regular ring ( i.e. For every $x\in K$ there exists $u\in U(R)$ such that $xux=x$) and $x\in K$. Then $x=ev$ for some $e\in Id(R)$ and $v\in U(R)$.

**Proof.** Let $x\in K$. Then for some $u\in U(K)$, $xux=x$. Thus $xu\in Id(K)$. Let v be the multiplicative inverse of u in K. Then $x=xuv$ a product of an idempotent and a unit of K.

**Theorem 3.** The ring R is $\pi$-regular if and only if for every $x\in R$ there exist $e\in Id(R)$, $u\in U(R)$, and $w\in Nil(R)$ such that $x=eu+w$.

**Proof.** Suppose R is $\pi$-regular. Let $x\in R$. By Theorem 1, $K=R/Nil(R)$ is regular. Thus K is unit regular by Theorem 2. By Lemma 4, $[x]=[c][u]$ for some $[c]\in Id(K)$ and $[u]\in U(K)$. By Lemma 3, there exists $e\in Id(R)$ such that $[e]=[c]$ in K and by Lemma 4 we have

$u\in U(R)$. Thus $[x]=[e][u]$ implying $x-eu=w$ for some $w\in Nil(R)$. Hence $x=eu+w$.

Let $x\in R$ such that $x=eu+w$ for some $e\in Id(R)$, $u\in U(R)$, and $w\in Nil(R)$. Then for some $n\geq 1$, $w^n=0$. Consider the expansion of $x^n=(eu+w)^n$. Observe that $(eu)^n=eu^n$ since $e\in C(R)$ and that the sum of the other terms are in $Nil(R)$ and of the form $ed$, $d\in R$, by semicommutativity. But $ed=e(ed)$, so replacing d by ed we can assume $d\in Nil(R)$. Hence $x^n=(eu+w)^n=eu^n+ed$ for some $d\in Nil(R)$. But $u^n+d=v\in U(R)$. Hence $x^n=ev$. Let r be the multiplicative inverse of v in R. Then $x^n r x^n=x^n$.

**Example 1.** Let K be a Principal Ideal Domain with 1, and m be a nonzero nonunit element of K. Then $m=p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_n^{\alpha_n}$. Where the $\alpha_i$'s$\geq 1$ and the $p_i$'s are distinct primes in K. Let $S=K/(m)$ and $I=(p_1p_2\ldots p_n)$ an ideal of K. Then $Nil(S)=I/(m)$. Thus $S/Nil(S)\approx K/I$ is isomorphic to a finite direct sum of fields. Hence $S/Nil(S)$ is regular. Thus S is $\pi$-regular.

**Example 2.** Let R be a semicommutative Artinian ring. Since $J(R)$ is nilpotent and $Nil(R)$ is an ideal of R, $Nil(R)=J(R)$. We know that $R/Nil(R)=R/J(R)$ is isomorphic to a finite direct sum of simple rings. But it is easy to see that a semicommutative simple ring is a division ring. Thus $R/Nil(R)$ is isomorphic to a finite direct sum of division rings. Thus $R/Nil(R)$ is regular and therefore R is $\pi$-regular.

**Example 3.** Let R be a finite semicommutative ring. Then R is Artinian and hence it is $\pi$-regular. In particular, all finite commutative rings are $\pi$-regular ring.

## Related Results.

A ring K is called (s,2)-ring [8] if every element in K is a sum of two units of K.

The following Theorem gives a characterization of all semicommutative (s,2)-rings.

**Theorem 4.** A $\pi$-regular ring R is (s,2)-ring if and only if every element in Id(R) is a sum of two units of R.

**Proof.** Suppose every element in Id(R) is a sum of two units of R. Let $x \in R$. Since R is $\pi$-regular, $x=eu+w$ for some $e \in Id(R)$, $u \in U(R)$, and $w \in Nil(R)$ by Theorem 3. Since $e=v+r$ for some $v, r \in U(R)$, $x=vu+ru+w$. Since $w \in J(R)$ and $ru \in U(R)$, $ru+w \in U(R)$. Thus x is a sum of two units of R. The second direction is obvious.

In [4], it has been shown that if K is a $\pi$-regular ring whose primitive factor rings are Artinian, and which does not have Z/2Z as a homomorphic image, then every element of K is a sum of two units in K. However, for a semicommutative $\pi$-regular ring R, we give an alternative proof of this fact.

**Corollary 2.** Let R be a $\pi$-regular ring such that $2=(1+1) \in U(R)$. Then every element of R is a sum of two units in R.

**Proof.** Let $e \in Id(R)$. Since $(1-2e)(1-2e)=1$, $1-2e \in U(R)$. Since $2 \in U(R)$, e is a sum of two units. Hence by Theorem 4 the claim is clear.

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**References.**

1. Chacron, M. " On Algebraic Rings", Bull. Austral. Math. Soc. 1(1969), 385-389.

2. Ehrlich, Gertrude, " Unit Regular Rings", Portugal. Math. 27 (1968) 209-212.

3. Ehrlich, Gertrude," Units and One-sided Units in Regular Rings", Trans. Amer. Math. Soc. 216 (1976), 81-90.

4. Fisher, Joe W.; Snider, Robert L., " Rings Generated by Their Units", J. Algebra 42(1976), no.2, 363-368.

5. Fisher, Joe W.; Snider, Robert L., " On The Von Neumann Regularity of Rings with Regular Prime Factor Rings", Pac. J. Math. 54 (1974) 135-144.

6. Goodearl K R, " Von Neumann Regular Rings", Pitman Publishing Limited (1979).

7. Henriksen, Melvin, " two Classes of Rings That are Elementary Divisor Rings", Arch. Math. (Basel) 24 (1973), 133-141.

8. Henriksen, Melvin, " Two Classes of Rings that Are Generated By Their Units", J. Algebra 31 (1974), 182-193.

9. Lam T. Y., " A first Course in Noncommutative Rings", Springer-Verlag, New York, Inc. (1991).

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